Fermions On One Or Fewer Kinks

Yi-Zen Chu[1] and Tanmay Vachaspati[1,2]
[1]*CERCA, Department of Physics, Case Western Reserve University, 10900 Euclid Avenue, Cleveland, OH 44106-7079,*
[2]*Institute for Advanced Study, Princeton, NJ 08540*

We find the full spectrum of fermion bound states on a Z_2 kink. In addition to the zero mode, there are int$[2m_f/m_s]$ bound states, where m_f is the fermion and m_s the scalar mass. We also study fermion modes on the background of a well-separated kink-antikink pair. Using a variational argument, we prove that there is at least one bound state in this background, and that the energy of this bound state goes to zero with increasing kink-antikink separation, $2L$, and faster than e^{-a2L} where $a = \min(m_s, 2m_f)$. By numerical evaluation, we find some of the low lying bound states explicitly.

I. INTRODUCTION

A novel feature of fermion-topological defect interactions is the appearance of fermion zero modes [1, 2, 3]. The existence of zero modes has important implications, leading to phenomena such as fractional quantum numbers [4] and superconducting cosmic strings [5]. In any physical setting, however, the system is expected to contain both defects and antidefects, and extended topological defects will frequently occur as closed structures, for example, closed loops of cosmic string, or closed branes in brane cosmology. Then it is important to determine the fate of a fermion zero mode in these situations.

The fate of fermion zero modes on topologically trivial structures, such as kink-antikink or cosmic string loop, has been addressed in Ref. [6]. The expectation that the fermion zero modes would be recovered as the kink-antikink separation, or the size of the cosmic string loop, is increased indefinitely, was not met in Ref. [6]. In the present paper, our primary aim is to reconsider the problem of fermions on kink-antikink backgrounds. Contrary to Ref. [6], we find that there are bound states on kink-antikink pairs whose energy vanishes exponentially fast with separation of the kink and antikink.

We start by finding all fermion bound states on a single kink. If $2m_f < m_s$ where m_f and m_s are the fermion and scalar masses, we find that the bound state spectrum only contains a zero mode. However, as we increase the fermion mass further, the number of bound states increases and is bounded by $2m_f/m_s$ as described in Sec. III. We then turn to the kink-antikink system, proving first that a bound state exists if the kink and antikink are well-separated. Our proof is based on a variational argument and allows us to obtain an upper bound on the energy of the bound state. The bound itself shows that the energy goes to zero with separation $(2L)$ faster than $\exp(-a2L)$ where $a = \min(m_s, 2m_f)$. Next, we evaluate the bound state energies numerically and confirm the exponential dependence on L. We also find an exponential decay of the ground state energy with increasing $2m_f/m_s$.

In the next section we set up the problem. We summarize our results in Sec. V. Identities involving hypergeometric function are included in the Appendix.

II. SETUP

The 1+1 dimensional field theory we are interested in is described by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi - g\phi\bar{\psi}\psi \quad (1)$$

where ϕ is a real scalar field, ψ is a two-component spinor, and the γ^μ are defined as

$$\gamma^t = \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ , \quad \gamma^z = i\sigma^1 = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \quad (2)$$

There are two masses in the model. The scalar mass is $m_s = \sqrt{2\lambda}\eta$ and the fermion mass is $m_f = g\eta$, where we are taking $g > 0$.

The Z_2 kink solution has the well-known form (*e.g.* see Ref. [3])

$$\phi = \eta \tanh\left(\frac{m_s z}{2}\right) \quad (3)$$

and the antikink is obtained simply by letting $z \to -z$. We shall also be interested in the system that contains a well-separated kink and antikink, for which the scalar field configuration can be chosen to be

$$\phi = \eta \tanh\left(\frac{m_s}{2}(z+L)\right) - \eta \tanh\left(\frac{m_s}{2}(z-L)\right) - \eta \quad (4)$$

The kink-antikink separation is $2L$.

Fermionic modes are found in the fixed scalar field background by solving the Dirac equation,

$$(i\gamma^\mu \partial_\mu - g\phi)\,\psi = 0, \quad (5)$$

where we will consider ϕ to be the kink solution of Eq. (3) and the kink-antikink configuration in Eq. (4). The modes will contain a set of bound states ($|E| < m_f$) and continuum states. In this paper, we will only be interested in determining the bound states with $E > 0$.

We write

$$\psi = e^{-iEt} \begin{bmatrix} (\beta_+ - \beta_-)/\sqrt{2} \\ (\beta_+ + \beta_-)/\sqrt{2} \end{bmatrix} \tag{6}$$

to get

$$(\partial_z + g\phi)\beta_+ \;=\; -E\beta_- \tag{7}$$
$$(\partial_z - g\phi)\beta_- \;=\; +E\beta_+ \tag{8}$$

Before proceeding further, it is convenient to perform a change to dimensionless variables defined by

$$z' \;=\; \frac{m_s z}{2}\ , \quad L' = \frac{m_s L}{2}\ , \quad E' = \frac{2E}{m_s}\ ,$$
$$g' \;=\; \sqrt{\frac{2}{\lambda}}g = \frac{2m_f}{m_s}$$

In what follows, we will drop the primes for notational convenience. The Dirac equations are then still given by Eqs. (7), (8), though with all variables having their dimensionless meanings, and the (rescaled) kink and kink-antikink backgrounds read

$$\phi_{\mathrm{K}} \equiv \tanh z \tag{9}$$

$$\phi_{\mathrm{K\overline{K}}} \equiv \tanh(z+L) - \tanh(z-L) - 1 \tag{10}$$

By substitution of one of Eqs. (7), (8) into the other, we obtain the 1-dimensional Schrödinger equations for $\beta_\pm$,

$$-\,\partial_z^2\beta_\pm + g(g\phi^2 \mp \partial_z\phi)\beta_\pm = E^2\beta_\pm, \tag{11}$$

allowing us to identify the potentials

$$V_\pm(\phi) \equiv g(g\phi^2 \mp \partial_z\phi) \tag{12}$$

Note that Eq. (11) actually contains two Schrödinger equations and the solutions of both must yield the same eigenvalue E^2.

The single kink (and antikink) backgrounds are odd functions of z, we see that under $z \to -z$, their first order equations transform into

$$-\left(\partial_z \pm g\phi\right)\beta_\pm = \mp E\beta_\mp. \tag{13}$$

That is, the parity reserved positive energy solutions are the parity un-reversed negative energy solutions. In other words, since kink and antikink are parity reversed functions of each other, the positive energy solutions on the kink are the negative energy solutions on the antikink; the negative energy solutions on the kink are the positive energy solutions on the antikink. Further, since the derivative of an odd function is an even function we observe that the corresponding Schrödinger equation, Eq. (11), is invariant under parity transformation: hence, if the energy eigenstates turn out to be non-degenerate (they are, as we will see below), they must be of a definite parity.

For even ϕ, the first order equations (7), (8) transform under parity $z \to -z$ into

$$\left(\partial_z \mp g\phi\right)\beta_\pm = \pm E\beta_\mp. \tag{14}$$

and hence $\beta_+(z) = \beta_-(-z)$. This includes the case of the kink-antikink background. An alternate way to see this is that $\partial_z\phi$ is an odd function of z, and the Schrödinger equation for $\beta_-(z)$ is identical to that for $\beta_+(-z)$. Hence if we have a solution to Eq. (11) for $\beta_+(z)$ for the kink-antikink background, $\beta_-(z) = \beta_+(-z)$ will be a solution for the β_- Schrödinger equation with the same value of E^2. In what follows, for the kink-antikink background, we will simply work with the β_+ equation.

III. FERMION BOUND STATES ON A KINK

We begin by solving the Schrödinger equation for a fermion on a single kink.

$$-\,\partial_z^2\beta_\pm + V_{K,\pm}(z)\beta_\pm = E^2\beta_\pm \tag{15}$$

where

$$V_{K,\pm}(z) \equiv g^2 - g(g\pm 1)\mathrm{sech}^2 z \tag{16}$$

For any value of $g > 0$, $V_{K,+}$ has the shape of a potential well with asymptotic maximum of g^2, and minimum value of $-g$ at $z = 0$. We know from quantum mechanics in 1 dimension that every non-positive potential that tends to zero asymptotically necessarily has at least one bound state. Hence $V_{K,+}(z)$ has at least one bound state for every g. Also, since $V_{K,+}(z)$ gets deeper with increasing g, we expect more and more bound states to appear with larger values of g. This expectation will be confirmed below. However, we also need a non-trivial bound state of the β_- Schrödinger equation which has the same energy eigenvalue as for β_+. Only then will $\beta_\pm$ solve the first order equations, Eq. (8), except if $E = 0$ for then we can take $\beta_- = 0$. For $0 < g \le 1$, $V_{K,-}$ is in the shape of a potential barrier and clearly has no bound states. This shows that for $0 < g \le 1$, the only possible bound state is with $E = 0$ and $\beta_- = 0$; the solution is

$$\beta_+^{(0)} = \mathrm{sech}^g z \tag{17}$$

More bound states do appear for $g > 1$ as we now find by explicit calculation.

Employing the prescription in Refs. [3, 7] we write

$$\beta_\pm = \mathcal{N}_\pm \mathrm{sech}^b z F_\pm(z) \tag{18}$$

with $b^2 = g^2 - E^2$, or $b = +\sqrt{g^2 - E^2}$, the positive choice of sign to ensure square integrability. Next we switch variables to

$$u \equiv \frac{1}{2}\left(1 - \tanh z\right) \tag{19}$$

and obtain the hypergeometric equation,

$$\begin{aligned} u(u-1)F''_{\pm}(u)+(b+1)(2u-1)F'_{\pm}(u) &+ \\ (b(b+1)-g(g\pm 1))\,F_{\pm}(u) &= 0 \end{aligned} \quad (20)$$

It can be inferred that the arguments of the hypergeometric function $F[\alpha_{\pm},\beta_{\pm};\gamma_{\pm};u]$ must be

$$\begin{aligned} \alpha_{\pm} &= b+\frac{1}{2}-\left(g\pm\frac{1}{2}\right) \\ \beta_{\pm} &= b+\frac{1}{2}+\left(g\pm\frac{1}{2}\right) \\ \gamma_{\pm} &= b+1 \end{aligned} \quad (21)$$

Observe that the $(g\pm 1/2)$ actually comes from taking a square root, so it ought to be contained within an absolute value sign, $|g\pm 1/2|$; but including α and β without the absolute value sign already covers both cases $g\pm 1/2>0$ and $g\pm 1/2<0$, since the hypergeometric function obeys the symmetry $F[\alpha_{\pm},\beta_{\pm};\gamma;u]=F[\beta_{\pm},\alpha_{\pm};\gamma;u]$.

The general solutions for $\beta_{\pm}$ are therefore

$$\begin{aligned} \beta_{\pm}(z) &= C_1\mathrm{sech}^b\, z\; F[\alpha_{\pm},\beta_{\pm};\gamma_{\pm};u] \\ &+C_2 e^{bz}F[\alpha_{\pm}-\gamma+1,\beta_{\pm}-\gamma+1;2-\gamma;u] \end{aligned} \quad (22)$$

As $z\to+\infty$, $\tanh z\to+1$ and from Eq. (A1) the hypergeometric function after the e^{bz} term goes to 1. As a result, we see that the second C_2 term becomes unbounded because of the e^{bz} factor. Hence we need to set $C_2=0$ for normalizability.

As $z\to-\infty$, we use the identity in Eq. (A2) to inform us that,

$$\begin{aligned} \lim_{z\to-\infty}\beta_{+}(z) &= \mathcal{N}_{+}\left(e^{bz}\frac{\Gamma[b+1]\Gamma[-b]}{\Gamma[g+1]\Gamma[-g]}\right. \\ &+ \left. e^{-bz}\frac{\Gamma[b+1]\Gamma[b]}{\Gamma[b+g+1]\Gamma[b-g]}\right) \end{aligned} \quad (23)$$

$$\begin{aligned} \lim_{z\to-\infty}\beta_{-}(z) &= \mathcal{N}_{-}\left(e^{bz}\frac{\Gamma[b+1]\Gamma[-b]}{\Gamma[g]\Gamma[1-g]}\right. \\ &+ \left. e^{-bz}\frac{\Gamma[b+1]\Gamma[b]}{\Gamma[b+g]\Gamma[b-g+1]}\right) \end{aligned} \quad (24)$$

The e^{-bz} term would be unbounded if its coefficient is finite. Recalling that the gamma function has poles at the negative integers and zero, we can then set the e^{-bz} term to zero by requiring that the argument of one of the gamma functions in the denominator to be a negative integer or zero. Since both $b+g$ and $b+g+1$ are strictly positive, we need

$$b_n^{\pm}-g+\frac{1}{2}\mp\frac{1}{2}=-n_{\pm}\in\mathbb{Z}^{-} \quad (25)$$

which implies

$$\begin{aligned} E_{n_+} &= \sqrt{n_+(2g-n_+)} \\ E_{n_-} &= \sqrt{(n_-+1)(2g-(n_-+1))} \end{aligned}$$

The solution for $\beta_{\pm}$ is valid only if their energy eigenvalues coincide, we get the additional requirement

$$n_+-n_-=+1 \quad (26)$$

The range of n_+ is determined by noting that $b_n^+=g-n_+$ from Eq. (25) and normalizability requires $b_n^+>0$. Therefore

$$0\le n_+<g \quad (27)$$

We then need to determine the relationship between the normalization constants $\mathcal{N}_{\pm}$ of these β_+ and β_- solutions by plugging them back into our first order equations (8). With some algebra involving the hypergeometric function identities (A3) and (A4), we can verify that our solutions do satisfy the first order equation provided we have

$$\frac{\mathcal{N}_+^{(n)}}{\mathcal{N}_-^{(n)}}=-\frac{E_n}{n} \quad (28)$$

where $n=n_+$ labels the n^{th} mode.

To summarize, on the kink background the positive energy fermionic bound states are given by

$$\begin{aligned} \beta_+^{(n)}(z) &= -\mathcal{N}_n\; E_n\mathrm{sech}^{g-n}z \\ F&\left[-n,2g-n+1;g-n+1;\frac{1}{2}\left(1-\tanh(z)\right)\right] \end{aligned} \quad (29)$$

$$\begin{aligned} \beta_-^{(n)}(z) &= \mathcal{N}_n\; n\;\mathrm{sech}^{g-n}z \\ F&\left[-n+1,2g-n;g-n+1;\frac{1}{2}\left(1-\tanh(z)\right)\right] \end{aligned} \quad (30)$$

$$E_n=\sqrt{n(2g-n)},\qquad 0\le n<g,\; n\in\mathbb{Z}^+$$

where we highlight that, because $-n$ and $-n+1$ are negative integers or zero, we see from (A1) the hypergeometric functions are really finite order polynomials in $u=(1-\tanh\; z)/2$.

$$\begin{aligned} F\left[-n,2g-n+1;g-n+1;u\right] &= \\ \sum_{m=0}^{n}\frac{(-n)_m(2g+1-n)_m}{m!(g-n+1)_m}u^m \\ F\left[-n+1,2g-n;g-n+1;u\right] &= \\ \sum_{m=0}^{n-1}\frac{(-n+1)_m(2g-n)_m}{m!(g-n+1)_m}u^m \end{aligned}$$

As an example, we can recover the bound state found in Ref. [6] by setting $n=1$,

$$\begin{aligned} \beta_+^{(1)}(z) &= -\mathcal{N}\sqrt{2g-1}\;\mathrm{sech}^{g-1}z\;\tanh z \\ \beta_-^{(1)}(z) &= \mathcal{N}\mathrm{sech}^{g-1}z \\ E_1 &= \sqrt{2g-1} \end{aligned} \quad (31)$$

where $\mathcal{N}$ is a normalization factor.

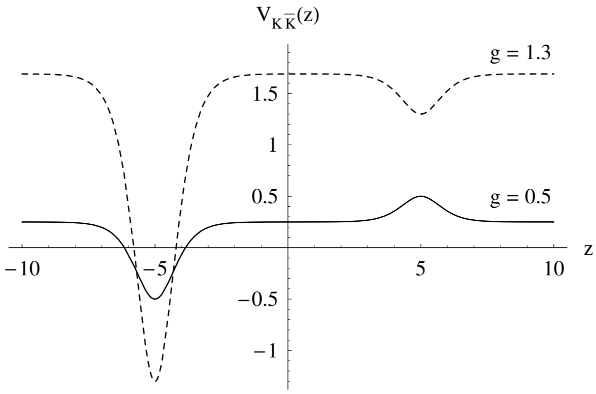

FIG. 1: Kink-antikink potentials $V_{\mathrm{K}\overline{\mathrm{K}}}$ for $g = 0.5$ and $g = 1.3$.

IV. BOUND STATES ON KINK-ANTIKINK

As discussed below Eq. (14), at the end of Sec. II, it is sufficient to find the solution for $\beta_+(z)$ in the kink-antikink background and then set $\beta_-(z) = \beta_+(-z)$. So we will only focus on finding β_+.

On inserting the kink-antikink background of Eq. (4), the Schrödinger equation (11) becomes

$$H_{K\overline{K}}\beta_+ \equiv \left(-\partial_z^2 + V_{\mathrm{K}\overline{\mathrm{K}}}\right)\beta_+ = E_n^2\beta_+ \tag{32}$$

where the potentials are

$$\begin{aligned} V_{\mathrm{K}\overline{\mathrm{K}}} \;&\equiv\; V_{\mathrm{K},+} + V_{\mathrm{K},-} - g^2 \\ &+\, 2g^2e^{-2L}\mathrm{sech}(z+L)\ \mathrm{sech}(z-L) \end{aligned} \tag{33}$$

where the expressions for $V_{\mathrm{K},\pm}$ are given in Eq. (16). The shape of this potential is illustrated in Fig. 1 for $g = 0.5$ and 1.3.

A. Proof of existence of bound states

There is a theorem by Simon [8] which states that a potential $\epsilon V(z)$ admits at least one bound state for all $\epsilon > 0$ if and only if $\int_{-\infty}^{\infty} V(z)dz \leq 0$.[1] Applying this criterion to our potentials (shifted by $-g^2$),

$$\int_{-\infty}^{+\infty}\left(V_{\pm}^{(\mathrm{K}\overline{\mathrm{K}})}(z) - g^2\right)dz = -4g^2 + 8g^2L\frac{e^{-2L}}{\sinh(2L)} \tag{34}$$

At large L, $8g^2Le^{-2L}/\sinh(2L)$ is small compared to $4g^2$, and hence the integral is negative. Solving for the zero of the right hand side amounts to solving

$$4L + 1 = e^{4L} \tag{35}$$

But $y = 4L + 1$ is the tangent line to $y = e^{4L}$ at $L = 0$. That is, the only solution to the above equation, and hence the only instance the integral of the potential becomes non-negative, is when $L = 0$. For all $L > 0$, therefore, we see that the kink-antikink background, as specified by Eq. (4), supports at least one fermion bound state for all non-zero values of the coupling g. Contrary to the claim by Postma and Hartmann [6], we see that spin does not pose any obstacle to the existence of fermion bound states on the kink-antikink.

B. A lowest energy upper bound

As mentioned in [6], the fermion zero mode ($E = 0$) solution on the kink-antikink is not normalizable, as can be verified by integrating (8) directly. That means E_0^2 is strictly positive. From the variational principle in quantum mechanics, we also know that the ground state energy E_0^2 is always less than or equal to the expectation value of the Hamiltonian $H_{K\overline{K}}$ with respect to an arbitrary square integrable wavefunction $|\psi\rangle$, namely,

$$E_0^2 \leq \frac{\langle\psi|H_{K\overline{K}}|\psi\rangle}{\langle\psi|\psi\rangle} \tag{36}$$

Motivated by the fact that

$$\varphi(z) \equiv \mathrm{sech}^g(z+L) \tag{37}$$

is the β_+ zero mode solution to a single kink at $z = -L$ and the only normalizable β_+ solution to the antikink at $z = +L$ is zero, we shall use φ as our trial wavefunction.

Inserting the Hamiltonian in Eq. (36) and using the equation obeyed by the zero mode state (Eq. (15) with $E = 0$) we get

$$\begin{aligned} 0 \;<\; E_0^2 &\leq \frac{\Gamma\left[g+\frac{1}{2}\right]}{\sqrt{\pi}\Gamma[g]}\int_{-\infty}^{\infty} dz\ \mathrm{sech}^{2g}z_+\ \mathrm{sech}\ z_- \\ &\times\ \left[-g(g-1)\mathrm{sech}\ z_- + 2g^2e^{-2L}\mathrm{sech}\ z_+\right] \end{aligned} \tag{38}$$

where we have denoted $z_\pm = z \pm L$ and also used the result [10, 11]

$$\int_{-\infty}^{\infty}\mathrm{sech}^{2g}z dz = \frac{\sqrt{\pi}\Gamma[g]}{\Gamma\left[g+\frac{1}{2}\right]} \tag{39}$$

The second term in the bracket in Eq. (38) gives a contribution proportional to

$$\begin{aligned} &2g^2e^{-2L}\int dz\ \ \mathrm{sech}^{2g+1}z_+\ \mathrm{sech}\ z_- \\ &\qquad < 8g^2e^{-4L}\int dz\ e^z\mathrm{sech}^{2g+1}z \end{aligned} \tag{40}$$

where we have used the inequality $\mathrm{sech}\ z_- < 2e^{z_-}$. The first term in the bracket also gives a contribution proportional to e^{-4L} for $g > 1$. However, for $0 < g < 1$, the

[1] An elementary proof by computing the expectation value of the Hamiltonian with respect to some trial wavefunction, can be found in [9].

contribution is estimated using

$$g(1-g)\int dz\ \ \mathrm{sech}^{2g} z_+\ \mathrm{sech}^2 z_- < \ g(1-g)2^{2g}e^{-4gL}\int dz\ e^{2gz}\mathrm{sech}^2 z \quad (41)$$

where we have used the inequality $\mathrm{sech}^{2g} z_+ < 2^{2g} e^{2gz_+}$.

The end result is

$$0 < E_0^2 < e^{-4L}\frac{\Gamma\left[g+\frac{1}{2}\right]}{\sqrt{\pi}\Gamma[g]} 8g^2 \int dz\ e^z \mathrm{sech}^{2g+1} z \qquad (42)$$

if $g > 1$, and

$$0 < E_0^2 < e^{-4gL}\frac{\Gamma\left[g+\frac{1}{2}\right]}{\sqrt{\pi}\Gamma[g]} g(1-g)2^{2g} \int dz\ e^{2gz} \mathrm{sech}^2 z \qquad (43)$$

if $0 < g < 1$ in the large L limit where the first term in Eq. (38) dominates over the second term.

These results provide an upper bound for the energy of the ground state in the kink-antikink background, the existence of which we proved in the previous subsection.

C. Numerical Solutions

We proceed to numerically solve the fermion bound state on the kink-antikink.

First we note that it is impossible for $\beta_\pm$ to both vanish at the same z. Recall that first order equations are solved uniquely by specifying one boundary condition for each β. So if it were the case that $\beta_+(z_0) = \beta_-(z_0) = 0$ for some z_0, then looking at (8), the unique solution is simply $\beta_+(z) = \beta_-(z) = 0\ \forall z$. In particular, we cannot have both $\beta_\pm$ go to zero at $z = 0$. As discussed earlier, since $\beta_+(z) = \beta_-(-z)$ for the kink-antikink we can thus set $\beta_\pm(z=0) = 1$ and rescale the solutions later if necessary.

The eigenvalues are written as $E_0 = \sqrt{|2g-1|\delta}$ and, for $n \geq 1$, $E_n = \bar{E}_n(1+\delta)$, with $\bar{E}_n \equiv \sqrt{n(2g-n)}$. They are searched for by solving (8) repeatedly with various values of δ, and watching the large $|z|$ asymptotic behavior of the solutions, as in the "shooting method". All of them eventually blow up, but as one tunes δ, the β_+ may say switch from going to negative infinity to going to positive infinity, as $z \to -\infty$. The exact eigenvalue lies between these two values of δ where this transition takes place, and the search for the eigenvalue primarily involves narrowing the gap between these two δs until the desired accuracy is achieved.

We selected $g = \pi$ and investigated how the energy levels near those of the single kink, $\sqrt{n(2\pi - n)}$, $n \in \{0, 1, 2\}$, are varied as the kink-antikink separation is altered from $L = 2.5$ thru $L = 7$. Referring to Fig. 2, one can infer that the first three energy levels roughly have an exponential dependence on the kink-antikink distance: $E_n \sim e^{-aL}$, for some $a > 0$ dependent on n. This indicates the $\{E_n\}$ approach that of their single kink counterparts as L is increased, in accordance with physical intuition.

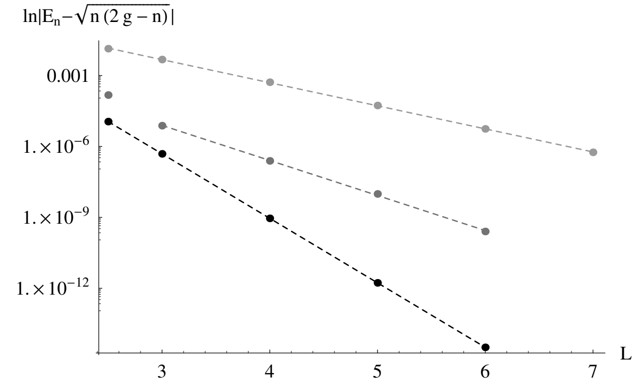

FIG. 2: Ground state and excited energy levels of fermion on kink-antikink near $\overline{E}_n = \sqrt{n(2g-n)}$, which are the energy levels on the single kink, for $g = \pi$. Here we plot the absolute value of the deviation from $\overline{E}_n$ to show, for the first three levels, the roughly exponential dependence on L, *i.e.* $\delta E_n \equiv |E_n - \overline{E}_n| \sim e^{-aL}$, with $a > 0$. From dark to light, the dots are for $n = 0, 1$ and 2, with best-fit slopes of -6.28, -3.41 and -2.25 respectively.

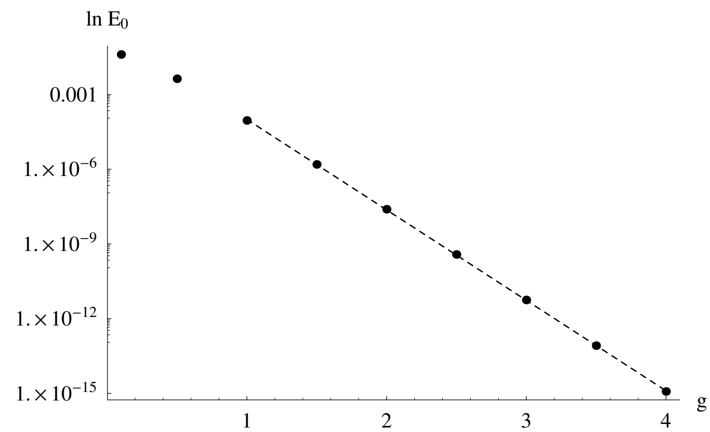

FIG. 3: Ground state energy vs. g, the Yukawa coupling, for $L = 5$. We see that $E_0 \sim e^{-8.36g}$.

For $L = 5$, we varied the coupling g from 0.1 thru 4 to examine the effect on the ground state energy eigenvalues. Fig. 3 provides evidence that the energies decrease roughly exponentially with increasing strength of the coupling.

The remaining figure, Fig. 4, shows the numerical β_+ solution to the kink-antikink system for the ground state of $\{g,\ L\} = \{0.1,\ 5\}$. It is compared against the corresponding analytic solution $\beta_+(z) = \mathrm{sech}^g(z+L)$ for the single kink at $z = -L$; the β_+ solution for the single antikink at $z = +L$ is zero. The numerical solution is normalized so that its approximate peak at $z = -L$ coincides with that of $\mathrm{sech}^g(z+L)$.

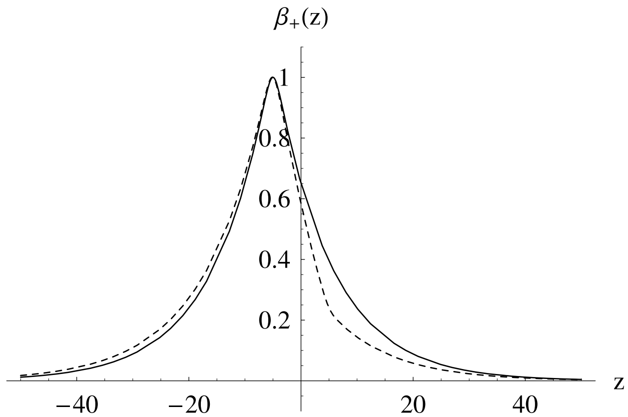

FIG. 4: Ground state of fermion on kink-antikink with $g = 0.1$, $L = 5$, and $E_0 \approx 0.04$. The solid line is the ground state $\beta_+ = \mathrm{sech}^g(z+L)$ solution on a single kink centered at $z = -L$. The dashed line is the numerical solution to the kink-antikink system.

V. CONCLUSIONS

We have tackled the problem of solving for bound states of the Dirac equation in (1+1) dimensions on kink and kink-antikink backgrounds. The resulting coupled first order equations can in turn be uncoupled to yield two Schrödinger equations, which we solve exactly for the single kink and antikink case. We find that the number of positive energy bound states on a kink is given by the smallest integer less than $g = 2m_f/m_s$. For fermions on a kink-antikink, we used the Schrödinger equations and results from non-relativistic quantum mechanics to prove that at least one bound state has to exist, for all non-zero values of the Yukawa coupling g. We then derived an upper bound for the lowest energy squared E_0^2 value which allowed us to prove that the ground state energy of the fermion on the kink-antikink tends to zero as the kink-antikink separation tends to infinity ($L \to \infty$). Appropriate boundary conditions for the first order equations were devised and employed to solve numerically the energy eigenvalues and eigenfunctions. For the specific examples we looked at, the lower lying bound states approached that of their single kink counterparts exponentially quickly as the kink-antikink distance was increased. Similarly, the ground state energy approached zero exponentially quickly as one increased the strength of the Yukawa coupling.

We expect our results to be valid also for the case of vortex-antivortex pairs, and for the case of loops of cosmic string. The lowest non-negative energy state on a loop of cosmic string will have positive energy that is suppressed by $\exp(-cR/w)$ where R is the radius of the loop and w is a width associated with the string and c is a numerical constant of order unity. In cosmological applications, this is an enormous suppression and we expect the picture derived on the assumption of exact zero modes to still hold true. Exceptions could occur if a loop shrinks and becomes small, or where a cusp occurs on a loop. For the case of superconducting strings [5], the small but non-zero energy of the lowest positive energy state means that charge carriers now have to jump from the Dirac sea to positive energy, requiring $2m$ energy, where m is the mass of the lowest positive energy state. An applied electric field with strength $< m^2/e$ along the string can cause this jump as in Schwinger pair production but the process is due to tunneling and is exponentially suppressed [15]. At stronger electric fields, the process would be unsuppressed. The critical value of the electric field for unsuppressed pair production is $\sim m_f^2 \exp(-cL/w)/e$ where e is the electric charge of the fermion.

Another setting where fermion zero modes are believed to play an important role is in brane cosmology where fermions are trapped on 3+1 dimensional branes in a higher dimensional bulk universe. If the fermions have zero modes in the brane background, it corresponds to massless fermions that are trapped on the brane and this is a possible explanation for massless standard model fermions living in a 3 dimensional space. In light of our results, if the brane can be thought of as a domain wall, in addition to the fermion zero modes, we may also expect other bound states to exist for a range of parameters. If the brane is closed or the bulk contains neighboring antibranes, the fermion zero modes will become bound states with an exponentially small mass. This may either be viewed as an undesirable feature of the particular brane system, or else may be viewed as a means to probe brane configurations in the bulk via the properties of standard model fermions.

Acknowledgments

We thank Marieke Postma for discussions. Y.-Z.C. thanks Dai De-Chang for discussing methods of solving (8) in the early stages of this paper. All numerical and some analytic work in this paper were performed with *Mathematica* [12]. This work was supported in part by the U.S. Department of Energy and NASA at Case Western Reserve University.

APPENDIX A: HYPERGEOMETRIC FUNCTION IDENTITIES

In this appendix we collect various hypergeometric identities [7, 10, 13, 14] used in this paper.

$$F[\alpha,\beta;\gamma;u] \;=\; \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{m!(\gamma)_m} u^m$$

$$(\sigma)_m \equiv (\sigma)(\sigma+1)\dots(\sigma+m-1), |u|<1 \quad \text{(A1)}$$

$$F[\alpha,\beta;\gamma;u] = \frac{\Gamma[\gamma]\Gamma[\gamma-\alpha-\beta]}{\Gamma[\gamma-\alpha]\Gamma[\gamma-\beta]}F[\alpha,\beta;1+\alpha+\beta-\gamma;1-u] + (1-u)^{\gamma-\alpha-\beta}\frac{\Gamma[\gamma]\Gamma[\alpha+\beta-\gamma]}{\Gamma[\alpha]\Gamma[\beta]} \times F[\gamma-\alpha,\gamma-\beta;1-\alpha-\beta+\gamma;1-u], \quad |\arg[u]|<\pi,\ |\arg[1-u]|<\pi,\quad \alpha+\beta-\gamma \neq 0,\pm1,\pm2,\ldots \tag{A2}$$

$$u\frac{d}{du}F[\alpha,\beta;\gamma;u] = \alpha\left(F[\alpha+1,\beta;\gamma;u] - F[\alpha,\beta;\gamma;u]\right) \tag{A3}$$

$$(\alpha+1-\beta)(1-u)F[\alpha+1,\beta;\gamma;u] = (\alpha+1-\gamma)F[\alpha,\beta;\gamma;u] + (\gamma-\beta)F[\alpha+1,\beta-1;\gamma;u] \tag{A4}$$

[1] C. Caroli, P.G. de Gennes and J. Matricon, Phys. Lett. **9**, 307 (1964).

[2] R. Jackiw and P. Rossi, Nucl. Phys. B **190**, 681 (1981).

[3] T. Vachaspati, *Kinks and Domain Walls: An Introduction to Classical and Quantum Solitons*, Cambridge University Press (2006)

[4] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).

[5] E. Witten, Nucl. Phys. B **249**, 557 (1985).

[6] M. Postma and B. Hartmann, arXiv:0706.0416 [hep-th].

[7] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part I and Part II, McGraw Hill (1953)

[8] B. Simon, Ann. Phys. (N.Y.) 97, 279–288 (1976)

[9] K. R. Brownstein, Am. J. Phys. 68 (2), February 2000

[10] N. N. Lebedev, *Special Functions and their applications*, Dover Publications, Inc. (1972)

[11] H. Hochstadt, *The Functions Of Mathematical Physics*, Dover Publications, Inc. (1986)

[12] Wolfram Research, Inc., Mathematica, Version 5.2, Champaign, IL (2005)

[13] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, as available at www.convertit.com/Go/ConvertIt/Reference/AMS55.ASP

[14] N. M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, John Wiley & Sons, Inc. (1996)

[15] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).